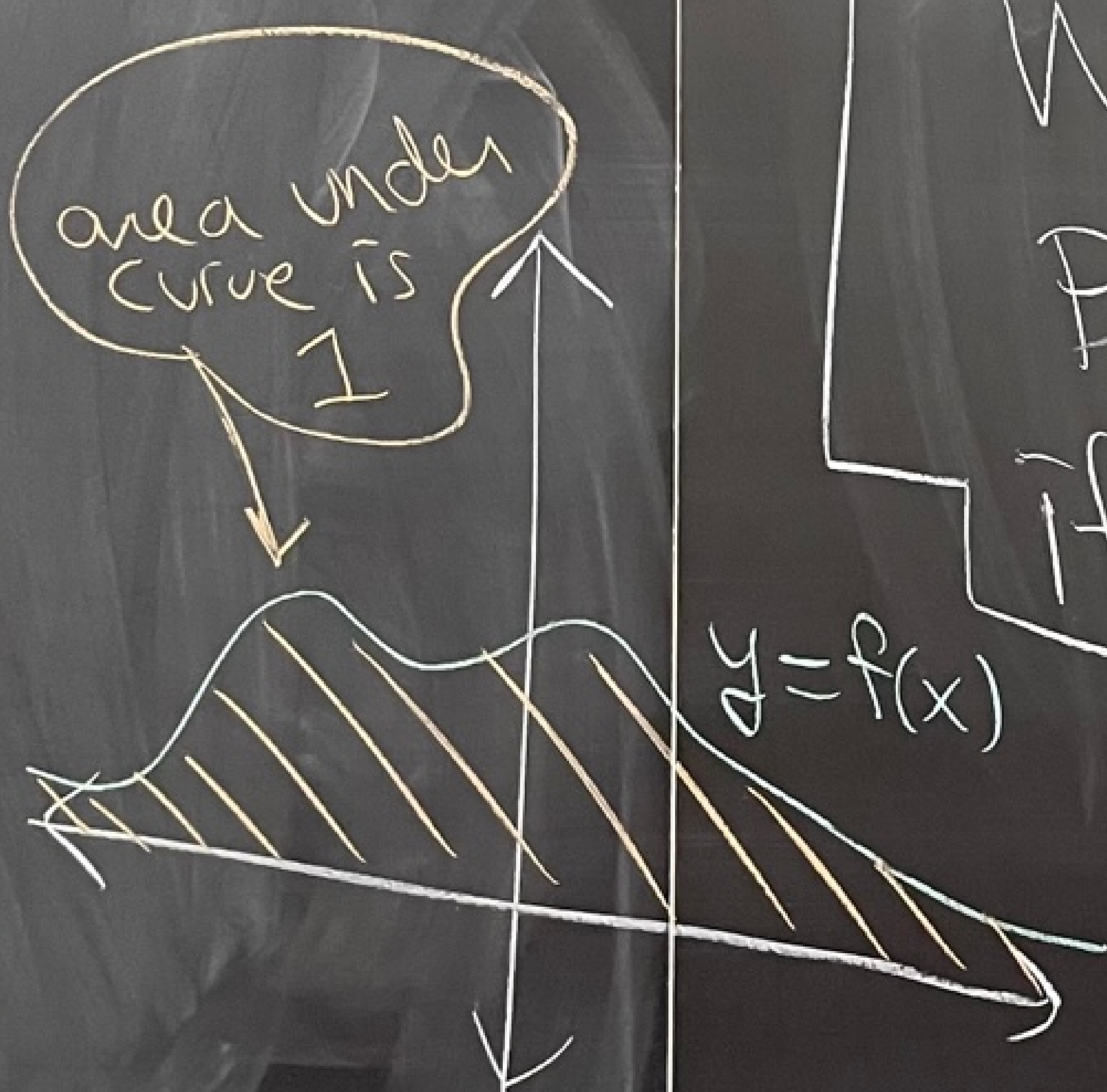
Not on final – HW 8 Topic

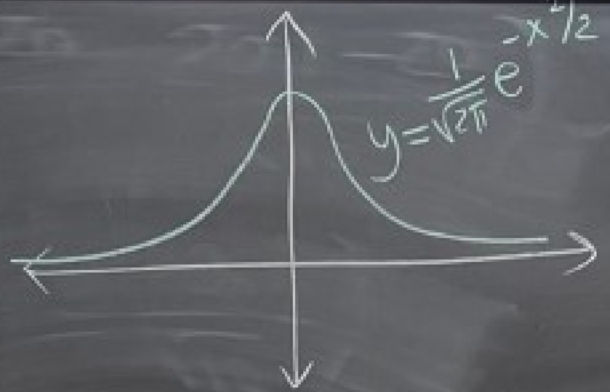
Continuous random Variables

Def: Let $f:\mathbb{R}\to\mathbb{R}$.
We say that f is a **probability density function** (p.d.f.) if

1. $f(x) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} f(x)\,dx$ exists and $\int_{-\infty}^{\infty} f(x)\,dx = 1$



Ex: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is a p.d.f.



① $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} > 0$ for all x

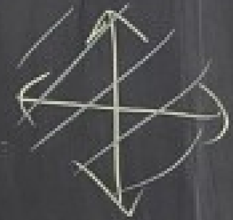
② Let $I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$

Then,

$$I^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx\right)\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy\right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy$$

We are integrating over the whole xy-plane.



polar coordinates

$r^2 = x^2 + y^2$

$dx\,dy = r\,dr\,d\theta$

$0 \leq r < \infty$

$0 \leq \theta < 2\pi$

over

e.

oordinates

x^2+y^2

$r\,dr\,d\theta$

$$= \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{-r^2/2}\, r\,d\theta\,dr$$

$$= \frac{1}{2\pi}\int_0^\infty \left[r\cdot e^{-\frac{r^2}{2}}\cdot \theta \Big|_{\theta=0}^{2\pi}\right] dr$$

$$= \frac{1}{2\pi}\int_0^\infty 2\pi\cdot r\cdot e^{-r^2/2}\,dr$$

$$= \lim_{t\to\infty}\int_0^t r e^{-r^2/2}\,dr = \lim_{t\to\infty}\left[-e^{-r^2/2}\right]_0^t$$

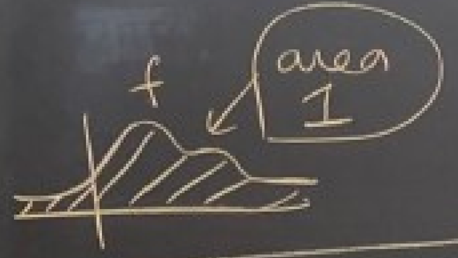
$$= \lim_{t\to\infty}\left[-e^{-t^2/2} - (-e^0)\right]$$

$$-e^{-t^2/2} \to 0$$

$$= 0 + 1 = 1.$$

$$I^2 = 1$$

So, $I = 1$.



$f:\mathbb{R}\to\mathbb{R}$

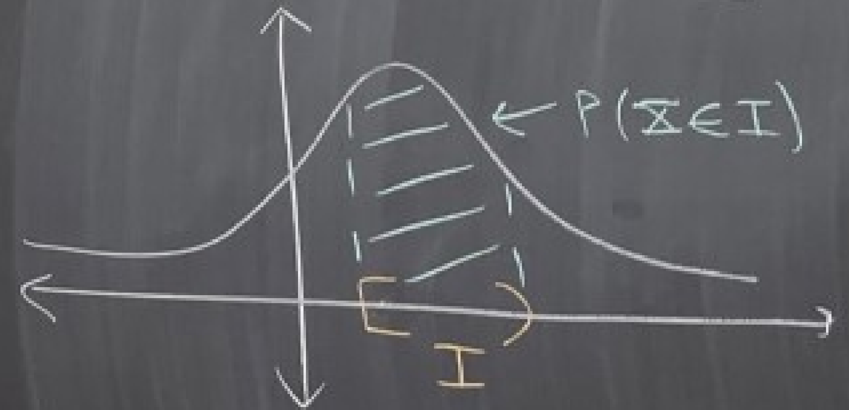
$f(x)\geq 0$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

More stuff not on test (HW 8)

Def: Let X be a random variable. We say that X is a <u>continuous random variable</u> if there exists a probability density function f where for any interval I in the real numbers we have

$$P(X \in I) = \int_I f(x)\,dx$$



So in particular

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

$$P(a \le X) = \int_a^\infty f(x)\,dx$$

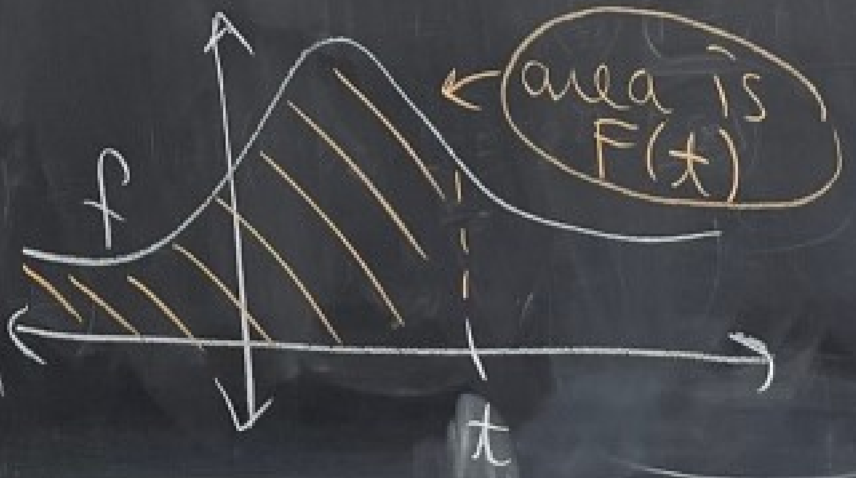
$$P(X \le b) = \int_{-\infty}^b f(x)\,dx$$

$$P(-\infty < X < \infty) = \int_{-\infty}^\infty f(x)\,dx = 1$$

The function f is called the probability density function (pdf) of X.

The cumulative distribution function of X (cdf) F is defined as

$$F(t) = P(X \le t) = \int_{-\infty}^t f(x)\,dx$$

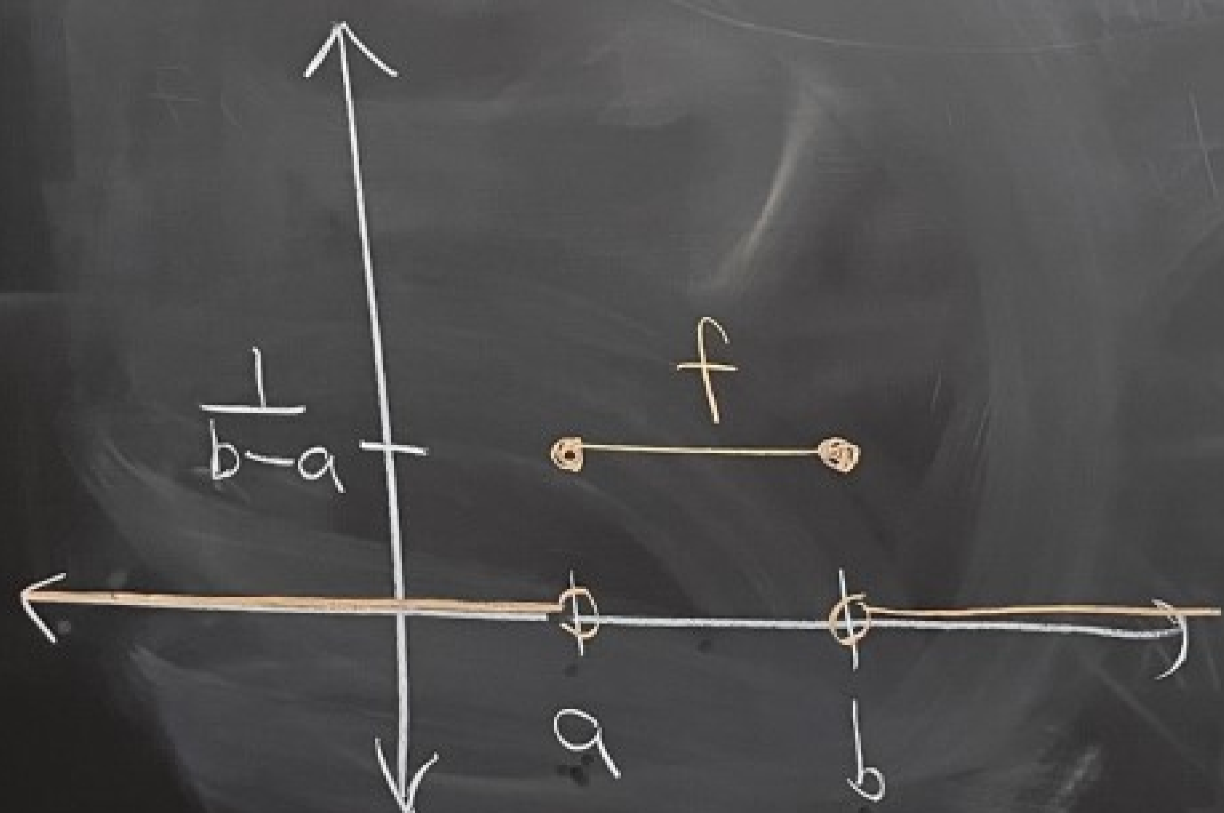


Ex: (The uniform distribution on $[a,b]$)

Let $a<b$.

Let

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

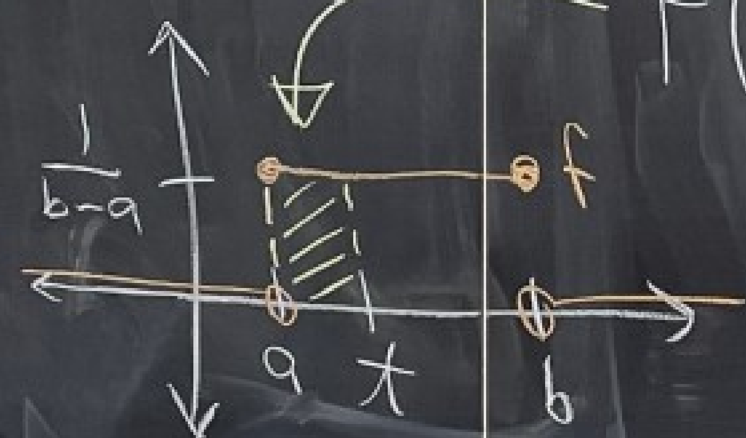


f is a pdf because

① $f(x) \geq 0$ for all x

② $$\int_{-\infty}^{\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \cdot (b-a) = 1$$



$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$

case 1: Suppose $t \leq a$.

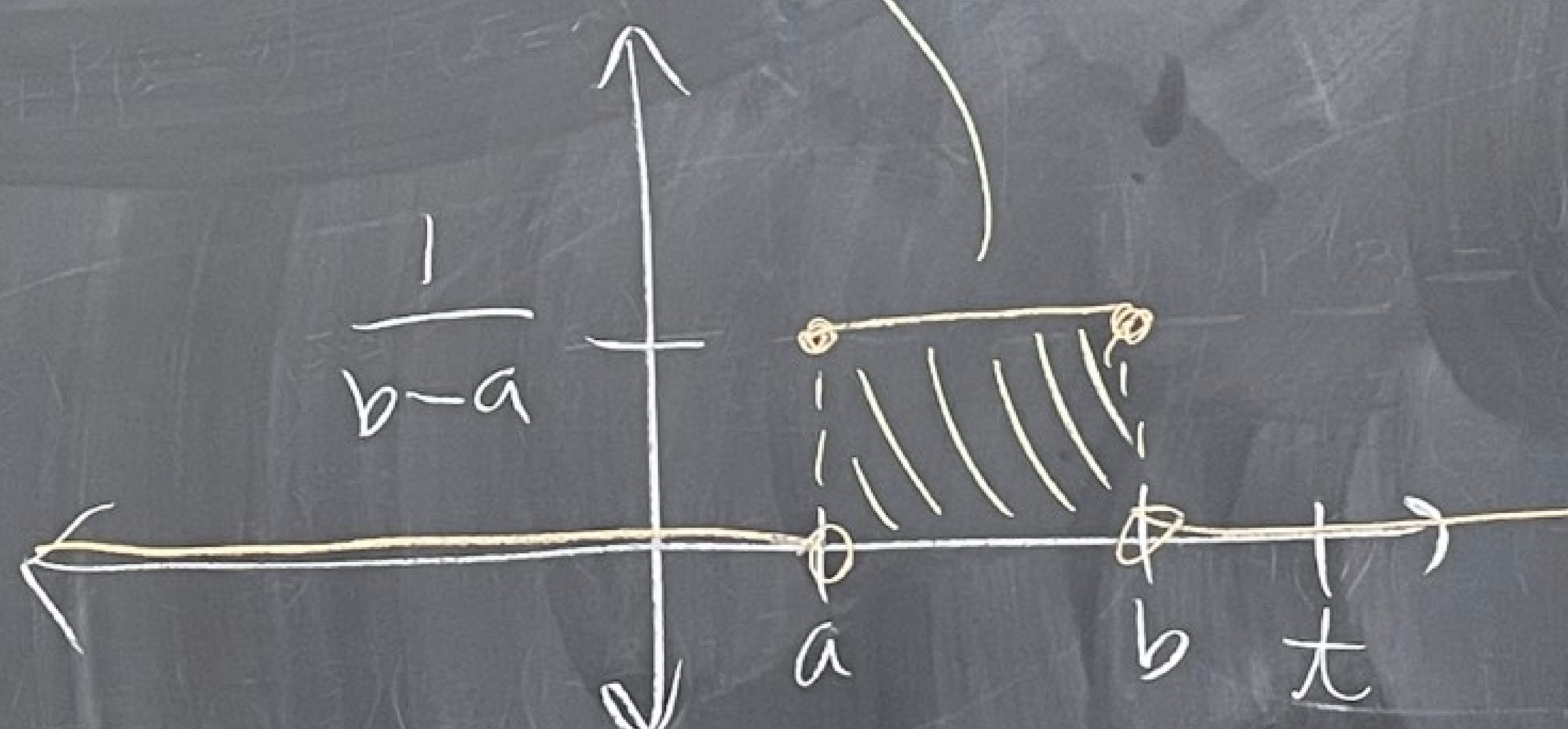
$$F(t) = \int_{-\infty}^{t} f(x)\,dx = \int_{-\infty}^{t} 0\,dx = 0.$$

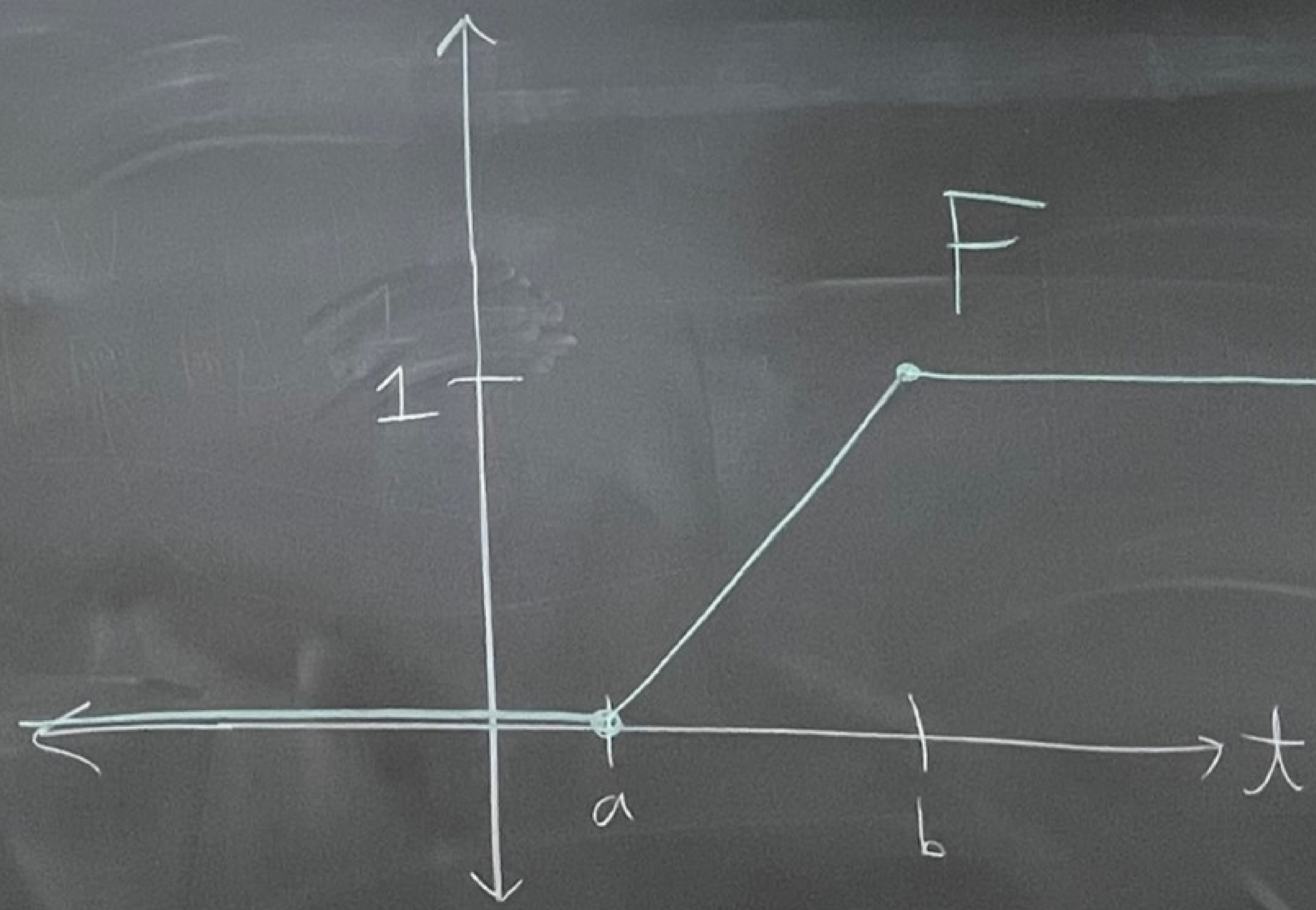
case 2: Suppose $a < t < b$.

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = \frac{t-a}{b-a}$$

<u>Case 3:</u> Suppose $b \le t$.

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = 1$$



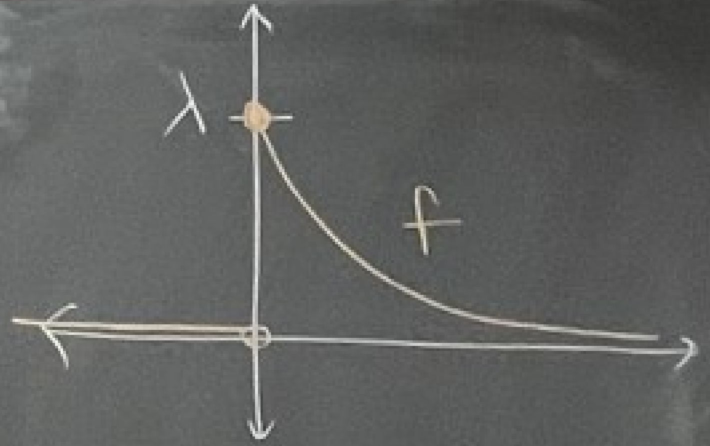

F
1
a
b
t

Ex: (Exponential random variable with parameter λ)

Let $\lambda > 0$.

Define

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



f is a pdf

① $f(x) \geq 0$ for all x

② $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \lim_{u \to \infty} \int_{0}^{u} \lambda e^{-\lambda x}\,dx$$

$$= \lim_{u \to \infty} \left[-e^{-\lambda x} \Big|_{x=0}^{u} \right]$$

$$= \lim_{u \to \infty} \left[-e^{-\lambda u} + e^{0} \right]$$

$$= \lim_{u \to \infty} \left[\frac{-1}{e^{\lambda u}} + 1 \right] = 0 + 1 = 1.$$

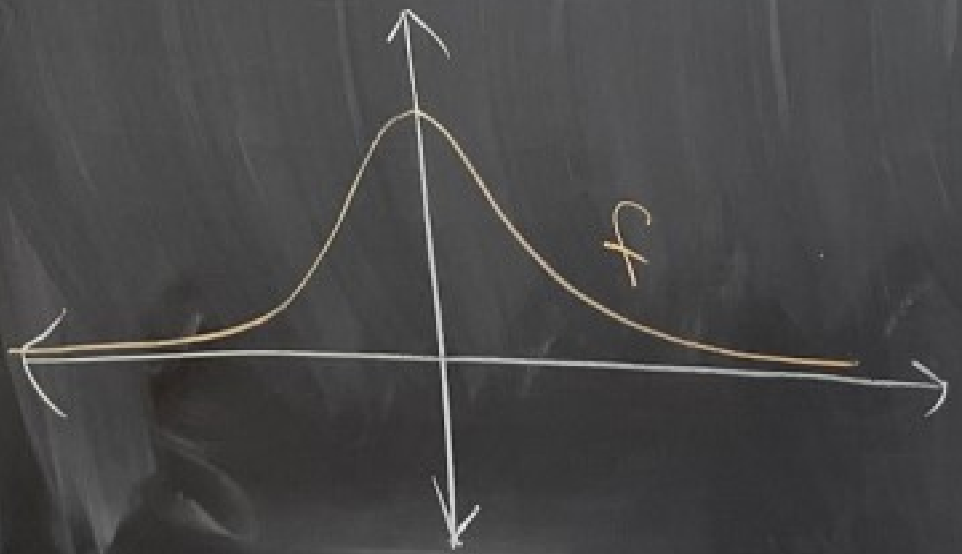
$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$

$$= \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Ex: (Standard normal distribution)

Let $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

We showed last week this is a pdf!



$$F(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$= \Phi(t)$$

We called this Φ

Def: Let X be a random variable with pdf f.

Then,

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

Let $\mu = E[X]$.

Then,

$$\mathrm{Var}(X) = \left(\int_{-\infty}^{\infty} x^2 f(x)\, dx \right) - \mu^2$$

$$\sigma = \sqrt{\mathrm{Var}(X)}$$